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A model of fatigue crack growth in polymers

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A model of fatigue crack growth is proposed based on a line plastic zone analysis. It is assumed that the effect of cycling is to reduce the craze stress to some proportion of the original value depending on the degree of unloading. Successive loadings result in growth of the craze with a corresponding increase in crack opening displacement. At some critical value of this displacement, crack growth occurs and the rate of growth is related to the applied stress intensity factor and the critical static value. The results of the model are applied to data on several polymers and a good description of growth rate, mean stress and frequency effects is given. Finally, some fatigue lives are predicted.

1. Introduction

There has been a considerable amount of published data on fatigue crack growth in polymers described in terms of the stress intensity factor K [1–5]. Most of it is reasonably well described by the well known Paris equation;

$$\mathrm{d}a/\mathrm{d}N \propto (\Delta K)^m \tag{1}$$

where $\mathrm{d}a/\mathrm{d}N$ is the crack growth per cycle, ΔK is the stress intensity amplitude, and m is a constant of approximately four. There is well established evidence of a mean stress effect, and frequency can also influence the crack growth rate [2, 3]. These effects are usually incorporated in the Paris equation by the addition of empirical parameters which give a reasonable description of the data.

Theoretical predictions of fatigue crack growth laws have also received much attention, particularly for metals. Since crack growth occurs under loadings which would not induce static fracture, it is necessary to introduce some form of either accumulation of damage or energy which results in eventual fracture. The line plastic zone model has been widely used for this purpose since it is an elastic analysis and leads to complete solutions [6, 7]. Various damage criteria have been used [8] but most of the effort has been in plastic energy accumulation analyses [7]. A visco-elastic extension of this latter approach has also been given which incorporates frequency effects [9].

Since most of the experimental data is rather similar in form for both metals and polymers, it is clear that the theoretical results will also be similar. It is thus difficult to judge the merits of the various assumptions in terms of the fit to experimental data and much of the theory has not been adequately tested.

In the light of this situation, it is necessary to justify presenting yet another analysis. The assumptions used here have the distinction of having been used to describe other crack growth phenomena in polymers with considerable success. It has been found that a constant critical crack opening displacement is an accurate description of visco-elastic crack growth phenomena over a wide range of time and temperature [10, 11] and the same criteria will be used here. In addition, environmental cracking and crazing in polymers has been accurately modelled [12] by assuming that the stress in the craze is constant but decreased by the environment. Here the same idea is used but now the decrease in craze stress is caused by the cyclic loading. These two simple assumptions may be incorporated into the line plastic zone analysis to give results which appear to model at least some data quite well. Thus, the

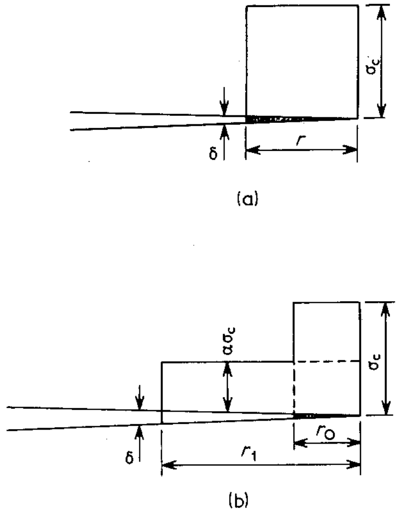

Figure 1 Line zone models;
(a) single stage zone, (b) two-stage zone.

$$\times\left(\alpha + (1-\alpha)\left\{\xi + \tfrac{1}{2}(1-\xi^2)\ln\left(\frac{1+\xi}{1-\xi}\right)\right\}\right)$$

where $\xi^2 = r_0/r_1$. For $\xi \ll 1$, this may be reduced to:

$$\delta = \frac{8\sigma_c}{\pi E}\left(\alpha r_1 + 2(1-\alpha)\sqrt{(r_1)}\sqrt{(r_0)}\right)$$

If $\delta = \delta_c$ at fracture, then

$$\delta_c = \frac{8}{\pi}\frac{\sigma_c}{E} r_c$$

where r_c is the critical single stage zone length at fracture, and we have

$$r_c = 2\sqrt{(r_1)}\sqrt{(r)} - \alpha r_1. \quad (4)$$

Thus, from Equation 4, the zone length at fracture r_1, is defined in terms of the constant r_c and the initial value r. The tip zone size at fracture may also be obtained

$$\frac{r_0}{r} = \frac{1}{(1-\alpha)^2}\left(1 - \alpha\frac{r_c}{r}\right). \quad (5)$$

analysis has the dual virtue of continuity with other fracture processes and simplicity, and for these reasons may prove useful.

2. The two-stage line plastic zone

Fig. 1a shows the conventional line plastic zone which has proved to be an accurate description of crack tip crazes in polymers. For small zones, i.e. $r \ll a$, the crack length, the crack opening displacement (COD) δ, and zone length r, are given by:

$$\delta = \frac{K^2}{E\sigma_c}, \qquad r = \frac{\pi}{8}\frac{K^2}{\sigma_c^2}$$

which may be rewritten as:

$$\sigma_c\sqrt{r} = \sqrt{\left(\frac{\pi}{8}\right)}K \text{ and } \delta = \frac{8}{\pi E}\sigma_c r \quad (2)$$

Fig. 1b is a similar small zone in which the stress is σ_c over a region of length r_0 at the zone tip, and $\alpha\sigma_c$ over the remaining length. The equivalent relationships to Equations 2 are

$$(1-\alpha)\sigma_c\sqrt{r_0}) + \alpha\sigma_c\sqrt{(r_1)} = \sqrt{\left(\frac{\pi}{8}\right)}K = \sigma_c\sqrt{(r)},$$

i.e.

$$\sqrt{r} = \alpha\sqrt{(r_1)} + (1-\alpha)\sqrt{(r_0)}, \quad (3)$$

and

$$\delta = \frac{8\sigma_c}{\pi E} r_1$$

3. The fatigue damage criterion

The craze at the crack tip in a stressed polymer is a system of interconnected voids in which the ligaments of material between the voids provide the strength of craze in the craze stress σ_c. If the craze is unloaded and then reloaded, it is reasonable to expect that some of the ligaments will be damaged thus reducing σ_c. In reality, several cycles would be necessary to complete this damage to a limiting value but here, for simplicity, we will assume that it occurs in a single unloading and reloading. It seems likely that the steady-state could be achieved rapidly but the solutions given here are unlikely to be precise for small numbers of cycles.

The degree of damage will depend on the amount of unloading and this may be described in terms of the limiting crack opening displacements during the fatigue cycle, i.e.

$$\delta_1 = \frac{K_{max}^2}{\sigma_c' E}, \qquad \delta_2 = \frac{K_{min}^2}{\sigma_c' E}$$

where σ_c' is some current value of the craze stress. We now suppose that the degree of damage may be defined in terms of the change in displacement $(\delta_1 - \delta_2)$ as a proportion of δ_1. Thus, the change in σ_c is given by:

$$\sigma_c\left(\frac{\delta_1 - \delta_2}{\delta_1}\right)f$$

where f is the damage factor. This may then be used to describe the damaged craze stress;

$$\alpha\sigma_c = \sigma_c - \sigma_c \frac{\delta_1 - \delta_2}{\delta_1} f,$$

so that

$$\alpha = (1-f) + fR^2, \qquad (6)$$

where $R = K_{min}/K_{max}$.

4. Crack incubation

Suppose that a crack is loaded with a maximum K value less than K_c, then a zone is formed with a craze stress σ_c and a displacement $\delta < \delta_c$ so that fracture does not occur. If K is now decreased to K_{min} and then reapplied, the craze stress on the original zone length r will be reduced to $\alpha\sigma_c$ so to maintain equilibrium some new craze must form with a craze stress of σ_c. From Equation 3, we have

$$\sqrt{r} = (1-\alpha)\sqrt{r_0} + \alpha\sqrt{r_1}$$

and we know that the length over which $\alpha\sigma_c$ acts:

$$r_1 - r_0 = r$$

in this case. Thus, the craze growth r_0 is given by:

$$\sqrt{r} = (1-\alpha)\sqrt{r_0} + \alpha\sqrt{(r+r_0)}$$

i.e.

$$\sqrt{r_0} = \sqrt{r_0}\frac{\sqrt{(1-\alpha)}}{1-2\alpha}[\sqrt{(1-\alpha)} - \alpha\sqrt{2}], \alpha \neq \tfrac{1}{2}.$$

This process may then be repeated with r_0 added to damaged zone length in the next cycle. In the general case, if the total zone length is r_N after N cycles, then the increase in length is given by:

$$\sqrt{r} = (1-\alpha)\sqrt{r_0} + \alpha\sqrt{r_N} + r_0$$

so that

$$\sqrt{r_0} = \frac{r}{1-2\alpha} \times \left\{(1-\alpha) - \alpha\sqrt{\left[1+(1-2\alpha)\frac{r_N}{r}\right]}\right\}, \quad \alpha \neq \tfrac{1}{2} \qquad (7)$$

and

$$\sqrt{r_0} = \sqrt{r}\left(1 - \tfrac{1}{4}\frac{r_N}{r}\right), \quad \alpha = \tfrac{1}{2}.$$

The zone length will increase by a decreasing amount each cycle and tend to a limiting value as $N \to \infty$ given when $r_0 \to 0$, i.e.

$$\frac{r_N}{r} = \alpha^{-2}. \qquad (8)$$

δ will also increase during this process and the critical condition is given by Equation 4

$$\left(\frac{K_c}{K}\right)^2 = \left(\frac{r_c}{r}\right) = 2\sqrt{\left(\frac{r_N}{r}\right)} - \alpha\left(\frac{r_N}{r}\right) \qquad (9)$$

Clearly, there is a limiting value of the ratio K/K_c, below which the fracture condition will not be achieved, given by:

$$(K/K_c)_{limit} = \sqrt{\alpha}. \qquad (10)$$

The number of cycles to reach δ_c for a given K/K_c for any α may only be found exactly by summing Equations 7, and Table I shows some computed values of r_N/r. Some of these data have been converted to K/K_c versus N to crack growth initiation values in Fig. 2. For low α values, the growth per cycle is small and Equations 7 may be approximated to a differential form in which $r_0 \equiv dr_1/dN$

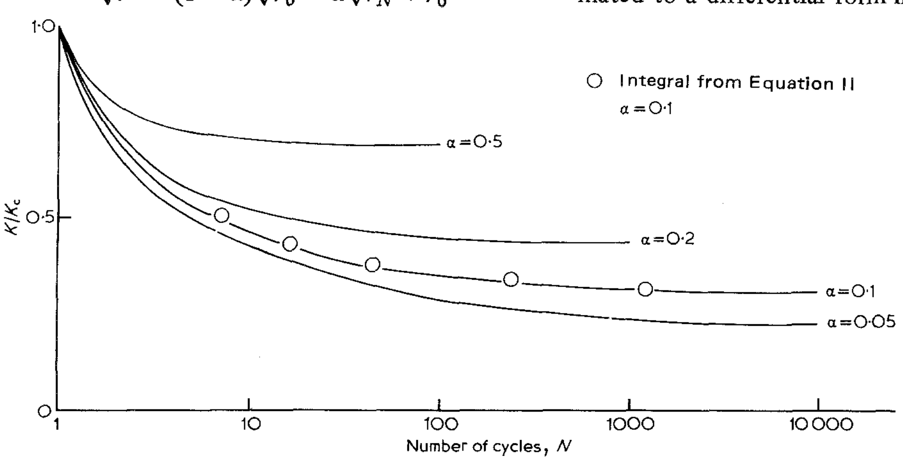

Figure 2 Number of cycles for crack growth initiation.

which, on integration, gives

$$N = \frac{2}{\alpha^2}\left\{\ln\left[\frac{\sqrt{(r/r_N)}-\alpha}{\sqrt{(r/r_N)}}\right] + \frac{\alpha}{\sqrt{(r/r_N)}-\alpha}\right\}. \quad (11)$$

This approximate result is also shown in Fig. 2 for $\alpha = 0.1$. The results for K/K_c are not valid for low cycle numbers because of the approximation inherent in Equation 4. No experimental data exists for these incubation cycles in polymers as far as the author is aware.

5. Crack propagation

When crack growth has initiated, a further cycle will result in the stress on r_0 falling to $\alpha\sigma_c$, but on reloading the crack will grow by r_0 as well as the zone increasing by that amount. Thus, $r_0 \equiv da/dN$

TABLE I r_N/r at various α and N values (Equation 7)

N	α				
	0.05	0.1	0.2	0.5	0.8
1	1	1	1	1	1
5	4.57	4.36	3.78	2.40	1.45
10	8.83	7.80	6.1	2.97	1.51
50	34.78	24.9	13.8	3.72	1.55
100	58.68	35.9	17.2	3.85	–
500	156.9	68.7	22.5	–	–
1,000	212.5	79.9	23.6	–	–
5,000	325.8	94.4	–	–	–
10,000	356.1	97.0	–	–	–
∞	400	100	25	4	1.56

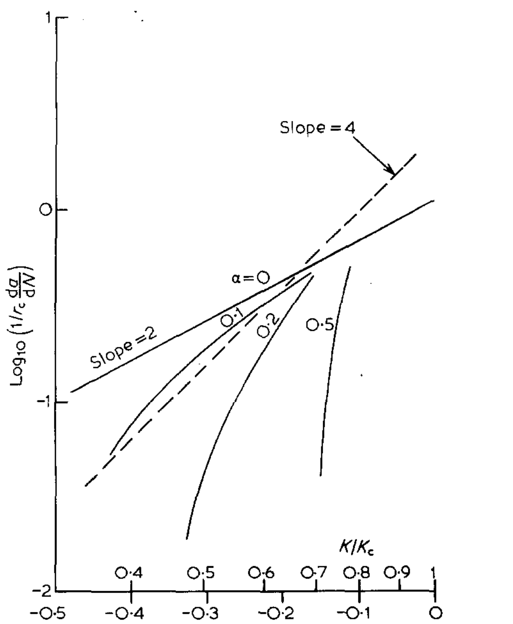

Figure 3 Fatigue crack growth rate as a function of K/K_c (log–log plot).

and Equation 5 may be written as:

$$\frac{1}{r_c}\frac{da}{dN} = \frac{1}{(1-\alpha)^2}\frac{K^2}{K_c^2} - \frac{\alpha}{(1-\alpha)^2}. \quad (12)$$

Fig. 3 shows Equation 12 plotted in the usual $\log K$ versus $\log da/dN$ form. The result for $\alpha = 0$ is, of course, a crack growth of r per cycle and gives a straight line of slope 2. For small α values (< 0.2), the high K values tend to this slope also, but at $K/K_c \simeq 0.5$ the slope is about 4 as in Equation 1. For lower values of K/K_c, there is a higher slope.

Some fatigue crack growth data for dry Nylon 66 taken from [13] is shown in Fig. 4 in the form of Equation 12. Four values of R were used and the points of each type were obtained on a single specimen. The predicted linear relationships are apparent but there are two regions with an abrupt transition. In each, the curves are accurately linear and may be extrapolated to the condition for $da/dN = 0$ which, from Equation 12, corresponds to $K^2 = \alpha K_c^2$. Fig. 5 shows these intercepts plotted as a function of R^2 in accordance with Equation 6 and a linear relationship results. At $R^2 = 1$, $\alpha = 1$ so that by extrapolation, the appropriate K_c values can be found, which are $K_{c1} = 3.61\,\text{MN m}^{-3/2}$ and $K_{c2} = 6.77\,\text{MN m}^{-3/2}$, with $f_1 = 0.72$ and $f_2 = 0.78$, respectively. Since α increases with R^2, the slopes would be expected to increase since the slope is

$$\frac{1}{(1-\alpha)^2}\frac{r_c}{K_c^2}$$

but they appear sensibly constant in Fig. 4. Using this slope in conjunction with Equations 2, the craze stress in the tip region σ_c may be calculated and some remarkably high figures are indicated, namely 7.2 and $3.4\,\text{GN m}^{-2}$, respectively. These are a direct consequence of the very small r_c values determined by the da/dN values, $1\,\mu$m. A plane stress and a plane strain fracture mode has been demonstrated to exist in Nylon 66 [14] and the appropriate K_c values are 3.5 and $8.5\,\text{MN m}^{-3/2}$ which correspond reasonably closely with those determined here. This would suggest that at low K values, the crack growth is dominated by plane strain while at higher values plane stress is effective.

Some rather more sparse data taken from [15] are shown in Fig. 6 for PMMA. Variations in slopes with R cannot be determined here and intercepts are measured by drawing parallel lines to the more

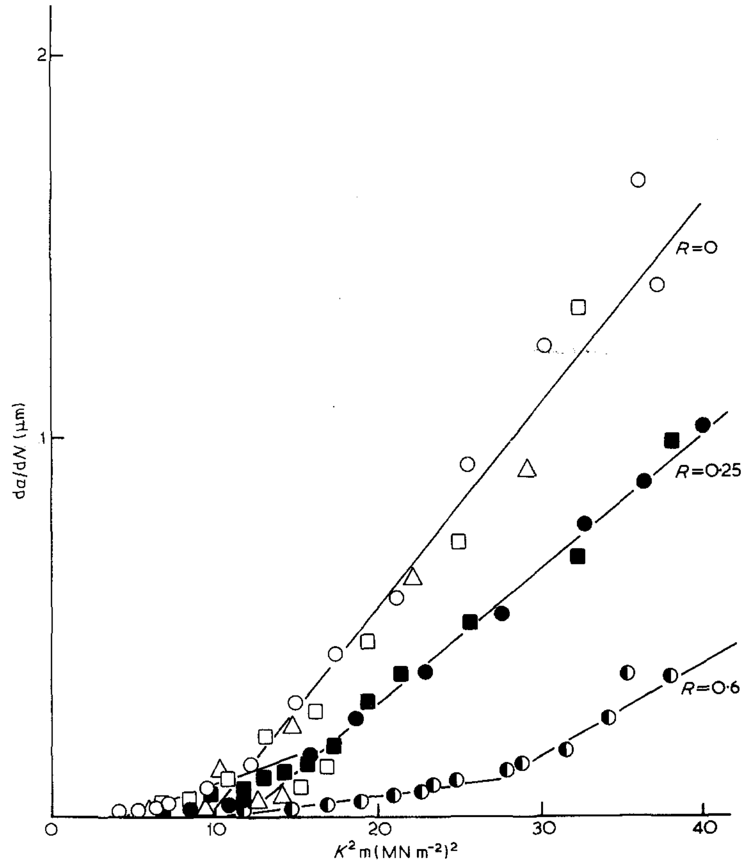

Figure 4 Fatigue crack growth data for dry Nylon 66 from [13] plotted in accordance with Equation 12.

plentiful low R value data. There is no evidence of any low K transition here, although a lower bound indicated by the broken line is possible. Fig. 7 shows the αK_c^2 data as a function of R^2 giving reasonable linearity with $K_c = 1.04\,\mathrm{MN\,m^{-3/2}}$ and $f = 0.82$. The craze stress in this case is much lower than the Nylon and is $735\,\mathrm{MN\,m^{-2}}$. This is still substantially greater than indicated by static tests.

Some similar data for polycarbonate, also from [15], is shown in Fig. 8 for low R values. The lower bound result at low K values with $\alpha \simeq 0$ is apparent here also. There is a craze stress of $325\,\mathrm{MN\,m^{-2}}$ for the major part of the curve.

Table II gives the collected data for these three materials and it is clear that σ_c is generally much higher than any yield or craze stress. The low K value data gives extremely high values which suggest no yielding at all, but the establishment of a state of triaxial stress in the heavily constrained crack tip region which is essentially still elastic. The fatigued craze stresses, $\alpha\sigma_c$, are of the same

TABLE II Derived values from crack propagation data

Material	K_c (MN m$^{-3/2}$)	σ_c (MN m^{-2})	α	$\alpha\sigma_c$ (MN m^{-2})	E (GN m^{-2})	δ_c (μm)	Comparison δ_c (μm)
Nylon 66	6.77	3,400	0.22	748	2.5	5.4	395 [14]
	3.61*	7,200	0.28	2,016	2.5	1.8	67 [14]
PMMA	1.04	735	0.18	132	2.7	0.5	1.7 [10]
PC	2.2†	325	0.09	29	3.0	5.0	6.6‡ [11]

*Plane strain values
†Taken from [11].
‡Converted from a plane stress value.

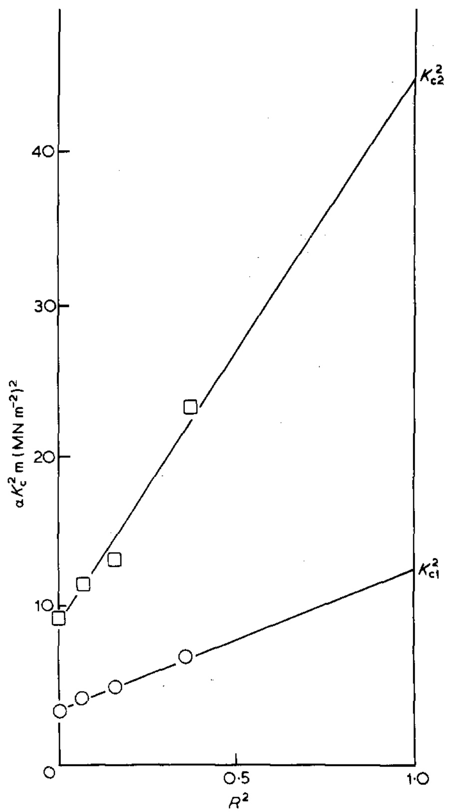

Figure 5 Intercepts at da/dN = 0 for Nylon 66 as a function of R^2.

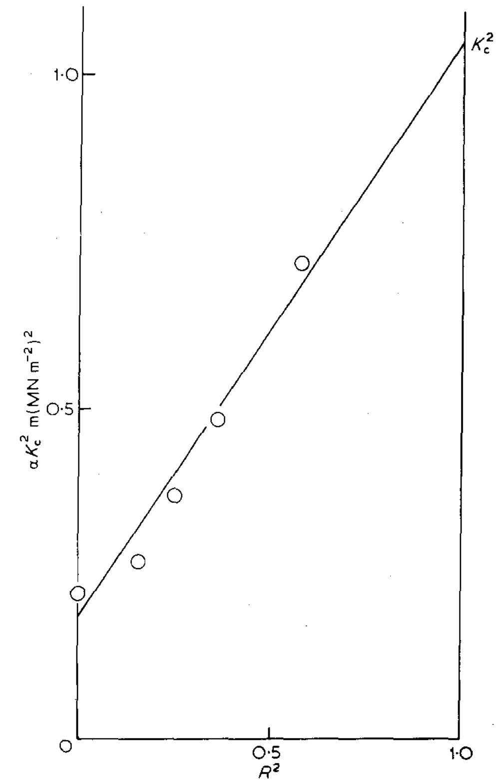

Figure 7 Intercept value at da/dN for PMMA as a function of R^2.

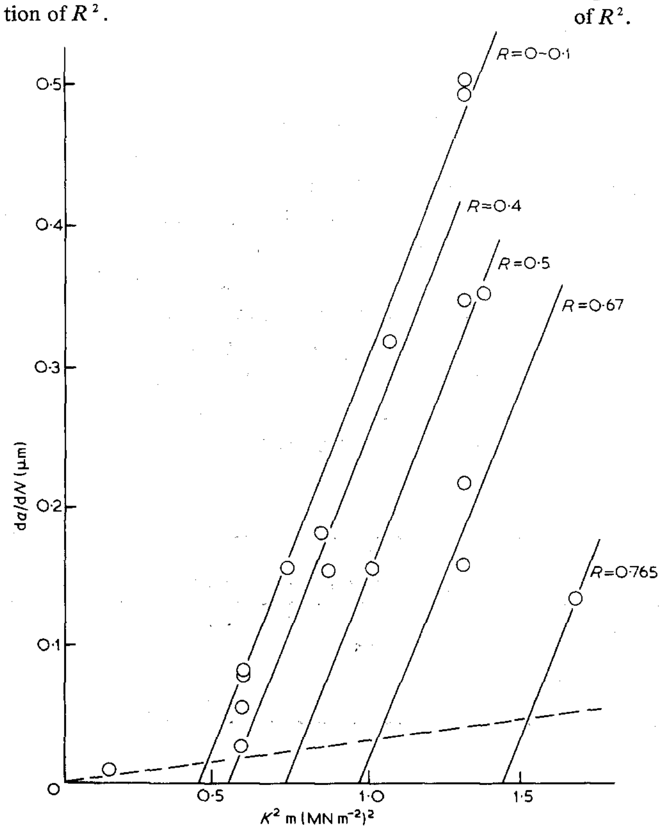

Figure 6 Fatigue crack growth data from [15], PMMA at 5 Hz.

order as expected craze stresses (although still very high for the low K data in Nylon). This suggests a mechanism in which the high stress concentration is reduced by fatigue to form the craze in addition to fatigue damage on existing craze material as originally assumed. The COD values are less than static values, particularly for Nylon 66. Data collected specifically for the purpose would be necessary to establish these values precisely.

TABLE III Frequency effect parameters

Material	m	$-mn$	n	$\tan\Delta$
Nylon 66	6.1	0.35	0.06 [13]	0.05 [13]
HDPE	3.1	0.53	0.17	0.06 [16]
PMMA	4.5	0.43	0.095 [15]	0.07 [10]

6. Frequency effects

There is considerable evidence available on frequency effects in the fatigue of polymers (see [1]). These effects may be deduced very simply from the model since the assumption of a constant COD means that K_c may be expressed in terms of σ_c and E, i.e. from Equation 2;

$$K_c^2 = \delta_c \sigma_c E.$$

It has been shown previously that a constant craze strain, e_y, is a reasonable assumption [10], so that

$$K_c = \sqrt{(\delta_c e_y)}E \quad (13)$$

Since the material is visco-elastic, E is time dependent and for many polymers this can be expressed as

$$E = E_0 t^{-n}$$

where E_0 is the unit time modulus, and n is approximately constant for any visco-elastic process and for low levels of visco-elasticity is approximately equal to the loss factor $\tan\Delta$. For cyclic stresses, the time scale which determines E is given approximately by

$$t = \omega^{-1}$$

where ω is the frequency, so that Equation 13 may now be written as

$$K_c = \sqrt{(\delta_c e_y)}E_0 \omega^n \quad (14)$$

Now, since δ_c is assumed constant, r_c is constant so that in Equation 12 changes in frequency will only change K_c for a given K value. If we now use the approximate form of Equation 12 as indicated by the Paris equation, we may write

$$da/dN \propto (K/K_c)^m \quad (15)$$

where m may be determined for a given K range (see Fig. 3). The dependence of da/dN on ω is then given from Equations 14 and 15, so that

$$da/dN \propto \omega^{-mn} \quad (16)$$

Fig. 9 shows data for dry Nylon 66 and high den-

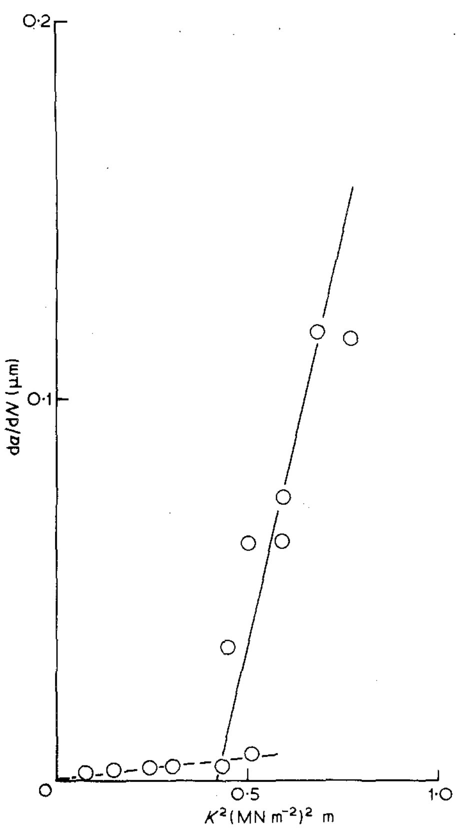

Figure 8 Fatigue crack growth data for polycarbonate from [15].

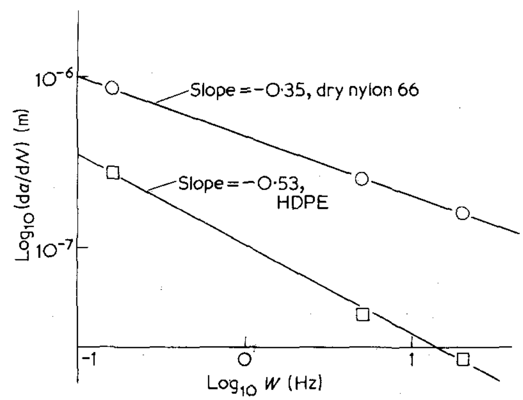

Figure 9 The effect of frequency on fatigue crack growth rate from [13].

sity polyethylene (HDPE) taken from [13] plotted as log da/dN versus log ω and reasonable linearity is indicated. Table III shows the slopes $(-mn)$ together with m values obtained from log K–log da/dN graphs. Also shown are results taken from [5] in which data on PMMA was fitted by a regression analysis to

$$da/dN \propto \omega^{-0.43} K_{mean}^{2.13} \Delta K^{2.39},$$

so that we may use

$$mn = 0.43 \text{ and } m = 2.13 + 2.39$$

The values n can be seen to agree reasonably well with those of tan Δ except for HDPE. This discrepancy probably arises from differences in the particular material used, since substantial variations are possible in HDPE.

7. Fatigue life

The fatigue life may be computed by integrating the crack growth rate until the applied value of K reaches K_c. If we consider the most simple case of a large sheet with a constant cyclic stress σ, and a small crack of length a, then

$$K = \sigma\sqrt{(\pi a)}.$$

From Equation 12, we may write:

$$\frac{da}{dN} = \frac{r_c}{(1-\alpha)^2}\frac{\sigma^2 \pi a}{K_c^2} - \frac{\alpha}{(1-\alpha)^2} \quad (17)$$

If the original crack length is a_0, then the original value of K is

$$K_0^2 = \sigma^2 \pi a_0,$$

and by rearranging and integrating Equation 17, we have

$$N_L = (1-\alpha)^2 \frac{K_c^2}{\sigma^2 \pi r_c} \int_{x_0}^{1} (x-\alpha)^{-1} dx$$

where $x = (K/K_c)^2$, and $x_0 = (K_0/K_c)^2$. On completing the integration, we have finally

$$N_L = \left(\frac{a_0}{r_c}\right)\left[\frac{(1-\alpha)^2}{x_0} \ln\left(\frac{1-\alpha}{x_0-\alpha}\right)\right] = \left(\frac{a_0}{r_c}\right)\phi \quad (18)$$

If we recall the number of incubation cycles, N_I, from Equation 11, we have a similar expression in terms of x_0, so that the total life is given by $N_L + N_I$. In general, N_I and ϕ are of similar magnitude, but for practical crack lengths, $a_0/r_c \gg 1$, so that $N_L \gg N_I$. Very small natural flaws could, however, result in a substantial proportion of the life being incubation.

Some failure lives and crack growth data are given in [13] for HDPE in a detergent Adinol. The curve from Equation 12 is shown in Fig. 10 in which the points are taken from the graphs in [13]. K_c is given as 2.2 MN m$^{-3/2}$ so that, from Fig. 10, we have $\alpha = 0.08$ and from the slope

$$r_c = 0.05\,\mu m$$

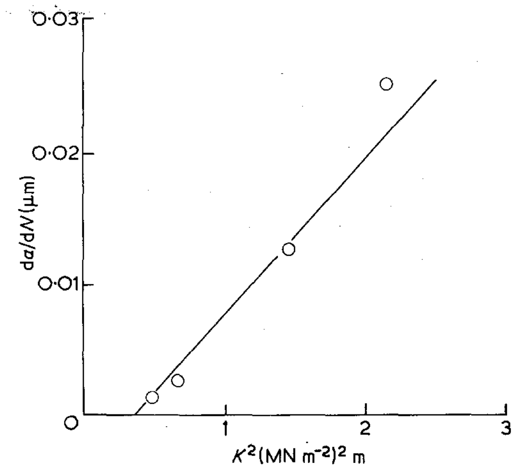

Figure 10 Fatigue crack growth data; HDPE in Adinol at 20 Hz.

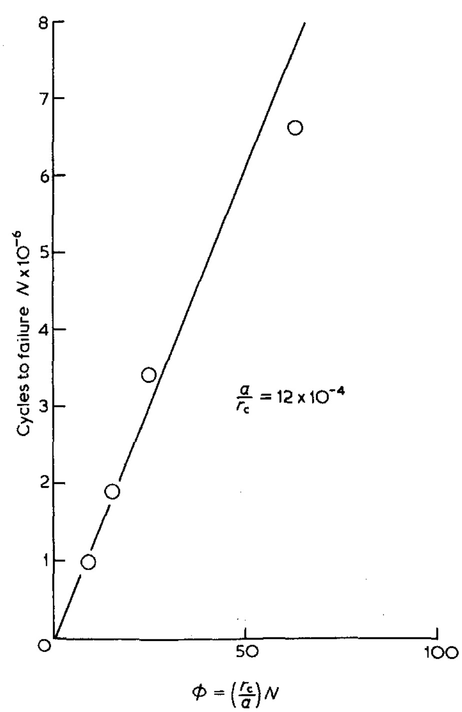

Figure 11 Cycles to failure as a function of ϕ for $\alpha = 0.05$; HDPE in Adinol.

The data given is for $a_0 = 9.5$ mm, which is sufficiently small to avoid finite width effects in the speciment of width 0.127 m, so that Equation 18 is valid. The number of cycles to failure are shown plotted in Fig. 11 versus ϕ for the appropriate K/K_c values and a good linear relationship is apparent. From this line, $a/r_c = 12 \times 10^4$ so that r_c may again be computed, i.e.

$$r_c = \frac{9.5 \times 10^{-3}}{12 \times 10^4} \text{ m} = 0.08\,\mu\text{m}$$

which agrees quite well with the value from the crack growth data.

8. Conclusions

The model appears to be successful in describing crack growth data on several polymers. The stress values at the craze tip regions are very high and suggest that the fatigue processes induce craze formation as well as damaging existing crazes. The general dependence of growth rate on K is in accord with the Paris equation and there is some evidence of a change of behaviour at low K values. In all the materials there is a tendency for the COD to be less in fatigue than in static failure. Frequency effects can be described accurately by the model by accounting for changes in K_c. Finally, fatigue life is accurately predicted by integrating the crack growth rate. The suggestions given here show some promise, at least under the range of conditions considered, and the model is worthy of further investigation.

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